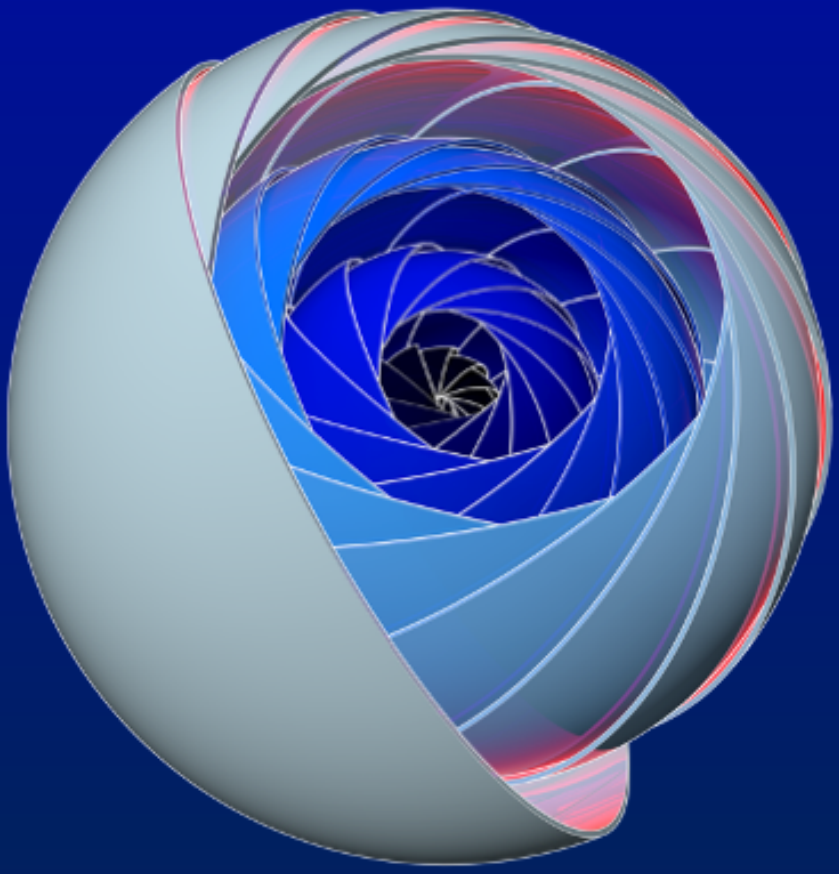
JOURNAL OF OPEN MATHEMATICAL PROBLEMS



AMR
ASSOCIATION FOR MATHEMATICAL RESEARCH

VOLUME 1 ISSUE 1

2025

JOURNAL OF OPEN MATHEMATICAL PROBLEMS

Founding Editors
George Andrews, Pennsylvania State University (USA)
Eric Friedlander, USC (USA)
Rob Kirby, University of California, Berkeley (USA)

Editor-in-Chief
Rob Kirby, University of California, Berkeley (USA)

Production Editor
Alex Kontorovich, Rutgers University (USA)

Editorial Board
Alain Connes, IHES (France)
Percy Deift, New York University (USA)
Simon Donaldson, Imperial College (UK)
Yakov Eliashberg, Stanford University (USA)
Mark Goresky, IAS (USA)
Carlos Kenig, University of Chicago (USA)
Richard Kenyon, Yale University (USA)
Stanley Osher, University of California, Los Angeles (USA)
Cheryl Praeger, University of Western Australia (Australia)
Kavita Ramanan, Brown University (USA)
Nicolai Reshetikhin, Tsinghua University (China)
Theodore Slaman, University of California, Berkeley (USA)
Daniel Tataru, University of California, Berkeley (USA)
Susanna Terracini, University of Turin (Italy)
Gunther Uhlman, University of Washington (USA)
Marcelo Viana, IMPA (Brazil)
Shmuel Weinberger, University of Chicago (USA)
Marie-France Vigneras, Institut de Mathématiques de Jussieu (France)
Lai-Sang Young, New York University (USA)

PUBLISHED BY

Association for Mathematical Research, amathr.org,
Davis, CA; Jenkintown, PA.

JOURNAL OF OPEN MATHEMATICAL PROBLEMS, 1(1):1–15, 2025
jomprob.org

Numbers in Ramsey theory

Fan Chung

Received 19 Sep 2024; Revised 27 Sep 2024; Accepted 20 Oct 2024

Abstract:
We survey several longstanding open problems on classical Ramsey numbers. These problems are easy to state but hard to solve while their history and developments are intimately associated with major advances in combinatorics.

Key words and phrases:
Ramsey Theory; Extremal Graphs; Quasi-Random

1 Introduction

The underlying philosophy of Ramsey theory can be described by the phrase "Complete disorder is impossible". If a sufficiently large structure (combinatorial, algebraic or geometric) is arbitrarily partitioned into finitely many parts, some patterns (substructures) must always be contained in one of the parts. Ramsey theory has its roots in the following classical results:

(1) For any partition of the k-element subsets of an infinite set S into finitely many classes, there is always an infinite subset of S with all its k-element subsets in one class. (Ramsey's Theorem [Ram30])

(2) In any partition of the integers into finitely many classes, some class always contains arbitrarily long arithmetic progressions. (van der Waerden's Theorem [vdW27])

(3) For any partition of the points in the plane into finitely many classes, some class always contains three points forming a right triangle of area 1. (Graham's Theorem [Gra80])

It is natural to consider finite versions of the above seminal theorems. Namely, one wishes to replace the notion of infinity by quantitative upper bounds that can be expressed as functions that depend on the numbers of parts and the sizes or shapes of the unavoidable patterns. This leads to a myriad of interesting

© 2025 Fan Chung
Licensed under a Creative Commons "Attribution-NonCommercial 4.0 International" license (CC BY-NC 4.0)
DOI: 10.56994/JOMP.001.001.001

problems that serve as the driving forces that have shaped up numerous areas in combinatorics. Many of these Ramsey-type problems are easy to state but notoriously difficult to solve, defying the attacks of many talented researchers. Here we will focus on unsolved problems on one of the oldest topics – the classical Ramsey numbers. Few of the classical Ramsey numbers are determined. Over the years, even incremental improvements of bounds for Ramsey numbers are cherished because of the accompanying new tools and methods. Computing small Ramsey numbers is no small feat but a test of the limit of computation (or providing a reminder of the deficiency of the power of computing). Many old and new results on Ramsey numbers serve as landmarks in the development of combinatorics and graph theory.

Ramsey theory is an area extraordinarily rich in problems. Here we mainly discuss selected few problems on classical Ramsey numbers without mentioning graph Ramsey, hypergraph Ramsey, Euclidean Ramsey problems as well as all the beautiful Ramsey-type problems in numerous interrelated areas, namely, on partitions of subsets, numbers, points in the plane or higher dimensional spaces.

2 Small Ramsey numbers

An introductory example to Ramsey numbers is the following 'party problem':

In a party of six people, there are three people who all know each other or all don't know each other.

Let a vertex represent a person and let an edge represent two persons who are friends. In a complete graph, denoted by K_n, on n vertices with $\binom{n}{2}$ edges, suppose we color the edges in 2 colors. For a fixed number k, how large should n be so that there are k vertices with all $\binom{k}{2}$ edges in the same color? Let $R(k)$ denote the least integer n such that any edge coloring of K_n contains a monochromatic K_k. The problem of interest is to determine the Ramsey number $R(k)$ for a given k.

Ramsey numbers and Ramsey theory are named after Frank Plumpton Ramsey who proved the infinite version of Ramsey theorem [Ram30] as well as showing finiteness of $R(k)$ in 1930. Erdős and Szekeres in 1935 rediscovered Ramsey numbers from the viewpoint of a famous geometric problem [ES35] (so-called "Happy Ending Theorem" on points in the plane avoiding forming small convex hulls).

For the case of $k = 3$, the upper bound $R(3) \leq 6$ can be proved basically by the pigeonhole principle. For the lower bound, a five cycle C_5 is the Ramsey graph for $R(3)$ as illustrated in Figure 1. A Ramsey graph G for $R(k)$ is a graph on $R(k)-1$ vertices without containing K_k and with independence number at most k (so the complement of G contains no K_k). Thus $R(3) = 6$.

For $k = 4$, in 1955 Greenwood and Gleason [GG55] proved that $R(4) = 18$ with the Paley graph P_{17} as the Ramsey graph. Note that a Paley graph P_q, for a prime power $q \equiv 1 \pmod 4$, is a graph on q vertices and edges $\{x, y\}$ with $x - y$ being a square in the finite field F_q. We remark that both the Ramsey graphs for $R(3)$ and $R(4)$ are Paley graphs and are unique (up to graph isomorphism) [Kal66].

For the case of $k = 5$, the best known upper and lower bounds are

$$43 \leq R(5) \leq 46. \tag{1}$$

The lower bound is due to Exoo [Exo89] in 1989 who gave a construction on 42 vertices that is an altered subgraph of a cyclic graph on 43 vertices. The upper bound is the result of the relentless and cumulative computational prowess over decades, with the best bound recently posted by Angeltveit and McKay

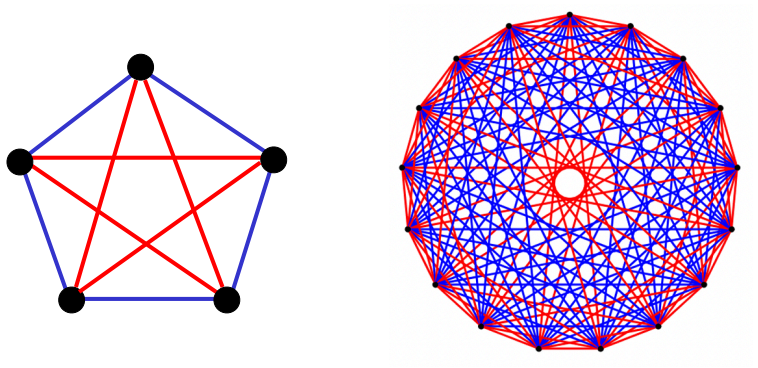


Figure 1: Colorings achieving the lower bounds of $R(3)$ and $R(4)$.

[AM24]. Motivated by the computational evidence, McKay and Radziszowski [MR97] conjectured that 43 is the value for $R(5)$.

Problem 1: Determine the Ramsey number $R(5)$. Is it true that $R(5) = 43$?

The quest of finding $R(5)$, a relatively small case, is a mystifying example that illustrates the limitation of our computational/theoretical ability at present. As an anecdote, Erdős [GS90] told the following story:

> *If aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five, we could marshall the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.*

Note that Erdős told this story in 1990 when the bound was $43 \leq R(5) \leq 55$ where the upper bound was due to Walker [Wal71] in 1971. As you can see, a great deal of improvements have been made over the years, see (1). For more values for small Ramsey numbers, the reader is referred to the dynamic survey [Rad24] for encyclopedic coverages on various Ramsey numbers.

3 The growth of Ramsey numbers

Let $R(k,t)$ denote the least integer n such the if the edges of K_n are colored in blue and red, there must be a blue K_k or a red K_t. For the symmetric case, $R(k,k)$ is also denoted by $R(k)$.

3.1 The diagonal Ramsey numbers

In 1947 Erdős [Erd47] gave the following estimate for $R(k)$,

$$\frac{1}{e\sqrt{2}} k 2^{k/2} < R(k) \leq \binom{2k-2}{k-1}. \tag{2}$$

The upper bound follows from the recurrence $R(k,l) \le R(k-1,l)+R(k,l-1)$ and therefore

$$R(k,l) \le \binom{k+l-2}{k-1} \tag{3}$$
$$R(k) = R(k,k) \le \binom{2k-2}{k-1} < \frac{4^k}{\sqrt{k}}$$

For the lower bound of $R(k)$, Erdős proved the existence of a good 2-coloring (i.e., containing no monochromatic K_k) without explicitly giving one. His proof literally gave birth to the area of combinatorial probabilistic methods. Here we give two short proofs that are slightly different from the original one in [Erd47], while the second proof gives an improvement of the lower bound by a constant of $\sqrt{2}$ by using the deletion method.

Proof #1:
Color edges of K_n with equal probability. The expected number of monochromatic K_k is $\binom{n}{k}2^{-\binom{k}{2}+1}$ which is less than 1 if $n \le \frac{1}{e\sqrt{2}}k2^{k/2}$.

Therefore there exists a good 2-coloring of K_n containing no monochromatic K_k. □

Proof #2:
Color edges of K_N with equal probability. Then delete a vertex for each monochromatic K_k. If we choose N close to $\frac{1+1/\sqrt{k}}{e}k2^{k/2}$, say within 1, then there exists a good coloring of K_n where

$$n = N - \binom{N}{k}2^{-\binom{k}{2}+1} \ge \frac{1}{e}k2^{k/2}.$$

Therefore we have proved

$$R(k) \ge \frac{1}{e}k2^{k/2}. \qquad \square$$

The best known lower bounds and asymptotic upper bound [Spe75, GNNW24] for $R(k)$ are:

$$\frac{\sqrt{2}}{e}k2^{k/2} < R(k) < (3.8)^k. \tag{4}$$

Thus the value of $\big(R(k)\big)^{1/k}$ is between $\sqrt{2}$ and 3.8.

Here are two oldest Ramsey problem that Erdős loved to mention in his numerous talks and ($\approx$ 1500) papers. For problems that he particularly wished to solve, Erdős often offered small monetary award for reflecting the level of difficulties and providing motivations for research.

Problem 2 ($ 100 [Erd47]):
Show that the limit

$$\lim_{k\to\infty} \big(R(k)\big)^{1/k} \quad \text{exists.}$$

Problem 3 ($ 250 [Erd47]):
Determine the value of

$$\lim_{k\to\infty} \big(R(k)\big)^{1/k} \quad \text{if the limit exists.}$$

The lower bound for $R(k)$ in (4), due to Spencer [Spe75] in 1975, is still the best lower bound for $R(k)$ after more than half a century. The improvement, just a factor of 2 from (2), was obtained by invoking Lovász local lemma, a major tool in probabilistic methods, which can be used to bound the probability of the intersection of a large number of 'sparsely' dependent events (i.e., the dependency graphs (or hypergraph) for the events have low degrees).

The upper bound in (4) was improved through a series of work:

• 1987 Graham and Rödl gave a proof of a weaker bound of order $\binom{2k}{k}/\log\log k$ in a survey paper [GR87] and mentioned Rödl's unpublished work of $R(k) \le c\binom{2k}{k}/(\log k)^c$.

• 1988 Thomason [Tho88] improved the lower bound by roughly a factor of $\sqrt{k}$, namely, $R(k) \le k^{-1/2+c/\sqrt{\log k}}\binom{2k}{k}$.

• 2009 Conlon [Con09] improved the lower bound by a superpolynomial but subexponential term in k, i.e., $R(k) \le k^{-c\log k/\log\log k}\binom{2k}{k}$.

• 2023 Sah [Sah23] proved $R(k) \le k^{-c(\log k)^2}\binom{2k}{k}$ by further optimized the techniques of Thomason and Conlon.

• In 2023, Campos, Griffiths, Morris and Sahasrabudhe [CGMS23] announced an exponential improvement showing $R(k) \le (4-\varepsilon)^k$ for $\varepsilon = 10^{-7}$ and sufficiently large k. The constant was further improved by Gupta, Norin and Wei [GNNW24] in 2024 by showing $R(k) \le (3.7992\ldots)^{k+o(k)} \le (3.8)^k$ for sufficiently large k.

All the above results for establishing Ramsey upper bounds involve a major theme called "quasi-randomness". The root of quasi-randomness can be traced to all the graph properties that were studied in extremal graph theory and random graph theory. Is there a natural way to organize or classify various graph properties? For example, for an integer k, the Ramsey property can be described as follows:

$\mathcal{P}_{\text{Ramsey}}(k)$: The graph G contains a clique of size k or an independent set of size k. Here a clique is a complete subgraph and the independent set is a clique in the complement of G.

Of course, this property can only make sense if $k \le 4\log n$ as shown in (2). Suppose we choose k to be a fixed integer and allow the number of vertices n of the graph to be large (or, we say, k is finite and independent of n). Then, indeed, much is known about $\mathcal{P}_{\text{Ramsey}}(k)$ via the theory of quasi-random graphs [CGW89]. When k is small, there is a class consisting of a dozen or so equivalent graph properties that are shared by random graphs. Any graph satisfying *any one* of the properties must of necessity satisfy *all* of the others. One of the properties for a graph G is the following property on enumeration/estimation of any graph on $k \ge 4$ vertices for a fixed integer k:

$\mathcal{P}_k(\varepsilon)$: for any graph H on k vertices, the number of occurrences of H in G is within a factor of $1+\varepsilon$ of what is expected (of a random graph).

Here by a random graph we mean that each pair of vertices is chosen to be an edge with probability $1/2$, (while the arguments can be modified for random graphs with some general edge density).

We say a property Q_1 implies a property Q_2 if for all ε there exists δ so that $Q_1(\delta)$ implies $Q_2(\varepsilon)$. For example, $\mathcal{P}_k(\varepsilon) \Longrightarrow \mathcal{P}_{\text{Ramsey}}(k)$ but the implication does not work in the opposite direction. Some useful properties in this family, denoted by $\mathcal{Q}$, of equivalent quasi-random properties [CGW89] include:

$\mathcal{P}_{4-cycle}(\varepsilon)$: for any graph H on k vertices, the number of occurrences of four cycles C_4 in G is within a factor of $1+\varepsilon$ of $n^4/16$.

$\mathcal{P}_{\text{eigenvalue}}(\varepsilon)$: The eigenvalues of G satisfy $|\lambda_1 - 1/2| < \varepsilon$ and $|\lambda_2| < \varepsilon$ where λ_1 and λ_2 denote the largest and the second largest eigenvalues (in absolute values) of the adjacency matrix of G .

$\mathcal{P}_{\text{discrepancy}}(\varepsilon)$: For any subsets S, T of the vertex set of G, each with at least εn vertices, the number of edges between S and T is within a factor of $1+\varepsilon$ of what is expected (of a random graph).

It is obviously true that $\mathcal{P}_k \Longrightarrow \mathcal{P}_{k-1}$. It is less obvious but true that $\mathcal{P}_{4-cycle} \Longrightarrow \mathcal{P}_k$ for any fixed $k \geq 4$. The eigenvalue property $\mathcal{P}_{\text{eigenvalue}}$ is very useful for validating explicit constructions as we will discuss later in Section 4. The discrepancy property $\mathcal{P}_{\text{discrepancy}}$ and its variations have often been used for proving upper bounds of Ramsey numbers in [Tho88, Con09, Sah23].

An example of quasi-random graphs in $\mathcal{Q}$ is the Paley graph which satisfies the eigenvalue property [CGW89] via Gauss sum and this immediately implies that, for instance, the occurrences of the Petersen graph in the Paley graph P_p is of order $2^{-15}\binom{p}{10}$. It is known that the size of maximum cliques in P_p, denoted by $f(p)$, satisfies

$$c \log p \log\log\log p \leq f(p) < \sqrt{\frac{p}{2}} + 1.$$

where the lower bound is due to Graham and Ringrose [GR90] in connection with the least non-quadratic residue problem that has been long studied. A slightly stronger lower bound $c \log p \log\log p$ is given by Montgomery [Mon71] assuming the generalized Riemann hypothesis. The constant of the upper bound was given by Hanson and Petridis [HP21]. Note that $f(q) \leq \sqrt{q}$ for a prime power q with equality holds if q is a square of a prime. It is known [BDR88] that the Paley graph P_{p^2} contains cliques of size p. The old conjecture by Vinogradov [Vin54], $f(p) \leq p^{\varepsilon}$ for any $\varepsilon > 0$, remains unsolved. Thus Paley graphs do not satisfy the Ramsey property $\mathcal{P}_{Ramsey}(k)$ for $k \approx c \log p$.

It is easy to see that the Ramsey properties satisfy $\mathcal{P}_{Ramsey}(k) \Longrightarrow \mathcal{P}_{Ramsey}(k-1)$ so the complexity gets harder when k grows. We can consider a hierarchy of stronger versions of the subgraph count property $\mathcal{P}_k$ where k is some slowly growing function of n, the number of vertices in the graph. Numerous questions can be asked. For example, are there any interesting or useful properties for a graph on n vertices that are equivalent to the property $\mathcal{P}_{\log n}$? Can we derive equivalence classes of these quantitative quasi-random properties (e.g., like $\mathcal{P}_{\log n}$) in a similar way as in [CGW89] for small k?

Almost all the results by Thomason [Tho88], Conlon [Con09], Sah [Sah23] and others [CGMS23, GNNW24] can be regarded as efforts to derive sharper estimates for some quantitative versions of quasi-randomness and then apply the results to improve upper bounds for Ramsey numbers. There is no question that our understanding of *quantitative quasi-randomness* plays a key role for improving Ramsey bounds and therefore remains a major challenge in graph theory.

3.2 Off-diagonal Ramsey numbers

Unlike diagonal Ramsey numbers for which any progress is extremely rare, the off-diagonal Ramsey numbers have been steadily advanced. Each new result in off-diagonal Ramsey numbers is often associated with new developments in probabilistic methods and is therefore a cause for celebration. The advancement in probabilistic methods can be well described by tracing the history of off-diagonal numbers $R(3,t)$ and $R(4,t)$.

For the upper bound for $R(3,t)$, the basic upper bound from (2) of order t^2 was improved by Ajtai, Komlós, Szemerédi [AKS80] in 1980 from a result on triangle-free graphs. They proved that a triangle-free graph on n vertices contains an independent set of size $c(n\log d)/d$ for some constant c where d is the average degree. Therefore $R(3,t) \le ct^2/\log t$. The method in [AKS80] can roughly be described as "coupled averaging" which is well captured in a short and elegant proof of Shearer [She83] who considered the following growth-rate function f, defined, for $d \ge 2$, by

$$f(d) = \frac{d\log d - d + 1}{(d-1)^2}, f(0) = 0, f(1) = 1/2.$$

Note that f satisfies the differential equation

$$(d+1)f(d) = 1 + (d-d^2)f'(d)$$

for $d \ge 0$ and it is easily checked that $f'(d) < 0, f''(d) \ge 0$, by choosing appropriate values of f' and f'' at 0 and 1. Now we want to choose a vertex v so that

$$(d+1)f(d) \le 1 + (dd_1 + d - 2d_1d_2)f'(d)$$

where d_1 is the degree of v and d_2 is the average of the degrees of the neighbors of v. This can be done since the average value of d_1d_2 over all vertices is the average value of d_1^2, which is at least d^2, and by taking average over all vertices v, the average of the left hand side of the above inequality is no more than that of the right hand side.

We wish to show for any triangle-free G, the independence number $\alpha(G)$ is at least $nf(d)$. We now remove such a v and its neighbors from G to form G' which is triangle free and, by induction, G' contains an independent set of size at least $(n-d_1-1)f(d')$ where $d' = (nd - 2d_1d_2)/(n-d_1-1)$. By substitutions and the fact that $f''(d) \ge 0$, we have

$$\begin{aligned}
\alpha(G) &\ge 1 + \alpha(G') \ge 1 + (n-d_1-1)f(d') \\
&\ge 1 + (n-d_1-1)f(d) + (n-d_1-1)(d'-d)f'(d) \\
&= 1 + (n-d_1-1)f(d) + (dd_1 + d - 2d_1d_2)f'(d) \\
&\ge nf(d).
\end{aligned}$$

As an immediate consequence, the constant in the upper bound for $R(3,t)$ is improved and therefore we have $R(3,t) \le (1+o(1))t^2/\log t$ which is still the best today.

For the lower bound of $R(3,t)$, in 1961 Erdős [Erd61] used classical probabilistic methods to get a lower bound $R(3,t) \ge ct^2/(\log t)^2$. The constant was steadily improved by, for example, using the Lovász Local Lemma. The breakthrough came in 1995 when Kim [Kim95] used semi-random methods to prove

$R(3,t) \geq ct^2/\log t$ which is of the same order as the upper bound. The constant of the lower bound was further improved to $1/4$ by Fiz Pontiveros, Griffiths and Morris [PGM20] and Bohman and Keevash [BK21], basically by differential equation methods.

The semi-random methods, sometimes called triangle-free process or differential equation methods, are basically as follows: At each step, choose a random edge (according to some distribution) and add the edge to the graph if no triangle is formed. After a number (to be determined) of steps, the resulting graph is triangle-free and, hopefully, still shares some properties of a random graph. While the basic step is simple, the analysis of the generated graph becomes technical because each step depends on all the previous ones. Intuitively and idealistically, one can try to anticipate how the expected value of a desired graph parameter (e.g., average degree of some types of subsets, density between subsets, subgraph counts, etc.) evolves when more edges are added or rejected. The easy part of the job is to write down a differential equation that should hopefully dictate the behavior of the expected value. The hard part is to make sure such a process does not diverge, oscillate or veer off somewhere else far from the differential equation. The triangle-free process by Kim [Kim95] has additional conditions to help guiding the evolution but the analysis becomes complicated. Bohman [Boh09] showed the above triangle-free process actually works for generating a Ramsey graph for $R(3,t) \geq cn^2/\log n$. The constant was further [PGM20, BK21] optimized to $1/4$. In 2025, Campos, Jenssen, Michelen and Sahasrabudhe [CJMS25] improved the constant to $1/3$ by using a variant of triangle-free process with a better 'seed' graph. Therefore, the best known bounds for $R(3,t)$ are

$$\big(\frac{1}{3}+o(1)\big)\frac{t^2}{\log t} \leq R(3,t) \leq (1+o(1))\frac{t^2}{\log t}.$$

Problem 4: Determine $\lim_{t\to\infty} \frac{R(3,t)}{t^2/\log t}$ if the limit exists.

Recently Mattheus and Verstraete [MV24] solved a $250 problem of Erdős by establishing the following lower bound for $R(4,t)$.

$$c_1\frac{t^3}{(\log t)^4} \leq R(4,t) \leq c_2\frac{t^3}{(\log t)^2}$$

for some constants c_1 and c_2. This lower bound improved the previous lower bound of order $t^{5/2}/(\log t)^2$ by Spencer [Spe75] via Lovász Local Lemma. Their method in [MV24], which is a combination of probabilistic methods and explicit constructions is quite different from the previous approaches and we will discuss it later in subsection 4. The upper bound for $R(4,t)$ can be proved by extending the ideas in [AKS80, She83]

$$R(4,t) \leq (1+o(1))\frac{t^3}{(\log t)^2}.$$

The next question is to improve the lower bound for $R(4,t)$.

Problem 5: Prove that

$$c\frac{t^3}{(\log t)^2} \leq R(4,t),$$

and determine the value

$$\lim_{t\to\infty}\frac{R(4,t)}{t^3/(\log t)^2} \text{ if the limit exists.}$$

For a fixed k, there are similar questions, such as the existence and evaluation of $\lim_{t\to\infty}\frac{R(k,t)}{t^{k-1}/(\log t)^{k-2}}$. We here omit the discussions and refer the reader to [Rad24].

3.3 The Ramsey numbers in many colors

Another direction in the study of classical Ramsey numbers is to let the number of colors (parts in a partition) vary. For a given k, what is the least number, denoted by $R_k(t)$ or $R(\underbrace{t,t,\cdots,t}_{k})$, so that any coloring of the edges of K_n in k colors must contain a monochromatic K_t for $n \geq R_k(t)$?

The party problem implies $R(3,3)=6$ and it is known [GG55] that $R(3,3,3)=17$. For $R(3,3,3,3)$, the upper bound 62 is due to Fettes, Kramer and Radziszowski in 2004, improving previous bound from 64. The lower bound [Chu73] remains at 51.

$$51 \leq R(3,3,3,3) \leq 62.$$

Problem 6: Determine the value of $R(3,3,3,3)$.

The value of $R(3,3,3,3)$ can be used to upper bound $R(\overbrace{3,3,\cdots,3}^{k})$ for $k>4$ by using the following recurrence [Chu74, CG98] and the current best upper bound 62 .

$$R(\overbrace{3,3,\cdots,3}^{k})-1 \leq 1+k\Big(R(\overbrace{3,3,\cdots,3}^{k-1})-1\Big) \leq k!\Big(\sum_{j=5}^{s}\frac{1}{j!}+\frac{R(3,3,3,3)-1}{4}\Big)$$

$$< k!\big(e-\frac{66-R(3,3,3,3)}{4}\big) \approx 2.5517\ k!$$

The best asymptotic lower bound for $R(\overbrace{3,3,\cdots,3}^{k})$ relies on the Schur number $S(k)$ which is the largest integer n so that integers $\{1,2,\ldots,n\}$ can be partitioned into k sum-free sets. By combining the recurrence $S(k+5) \geq 380S(k)+148$ in [ACP+22] and the close relation between Ramsey numbers and Schur numbers

$$R(\overbrace{3,3,\cdots,3}^{k}) \geq S(k)+2,$$

one can obtain

$$\big(R(\overbrace{3,3,\cdots,3}^{k})\big)^{1/k} \geq (380)^{1/5} \approx 3.2806\ldots$$

Problem 7($250 [CG98]) : Determine the value of $\lim_{k\to\infty}\big(R(\overbrace{3,3,\cdots,3}^{k})\big)^{1/k}$.

The value is currently between 3.2806 and infinity. Note that $R(\overbrace{3,3,\cdots,3}^{k})$ is supermultiplicative in k so the above limit exists [Chu73].

4 Combining probabilistic and constructive methods

Erdős established the lower bound [Erd47] for Ramsey number $R(k)$ by proving the *existence* of Ramsey graphs without giving *explicit constructions*. This leads to a puzzling scenario similar to the following problem:

> *Almost all real numbers are normal, (i.e., each digit appears asymptotically equally often in any base b expansion). Can you name a normal number? Is π normal? Is e normal? No one can answer these questions.*

In spite of numerous attempts, the best constructive lower bound for Ramsey numbers $R(k)$ is by Frankl and Wilson [FW81]. (Note that we will not discuss the study of constructing Ramsey graphs using deterministic Turing machine in polynomial time.) Here, by explicit constructions we mean succinct and finite descriptions, such as the construction of Paley graph, which however is not a Ramsey graph.

Frankl and Wilson's construction is based on two beautiful theorems [FW81] on set intersections. The first intersection theorem is by Ray-Chaudhuri and Wilson [CW75] and the second is due to Frankl and Wilson [FW81].

Intersection Theorem #1: [CW75] If $\mathcal{F}$ is a family of r-sets of integers $\{1,2,\ldots,m\}$ satisfying the following intersection property : If for all $F,F' \in \mathcal{F}, F \neq F'$, we have $|F \cap F'| \in S$ for some S with $|S| = s$, then $|\mathcal{F}| \leq \binom{m}{s}$.

Intersection Theorem #2: [FW81] If $\mathcal{F}$ is a family of r-sets of integers $\{1,2,\ldots,m\}$ satisfying the following intersection property : If for all $F,F' \in \mathcal{F}, F \neq F'$, we have $|F \cap F'| \not\equiv k \pmod r$ for a prime power r and some k, then $|\mathcal{F}| \leq \binom{m}{r-1}$.

We now consider a graph G having vertex set $\{F \subseteq \{1,2,\ldots,q^3\} : |F| = q^2-1\}$ for a prime power q and edge set $E = \{\{F,F'\} : |F \cap F'| \not\equiv q-1 (\mathrm{mod}\ q)\}$. The number of vertices in G is $n = \binom{q^3}{q^2-1}$. The second intersection theorem implies that the size of a clique is at most $k = \binom{q^3}{q-1}$ and the first intersection theorem implies that the size of an independent set is at most $k = \binom{q^3}{q-1}$ and $k \approx e^{c(\log n \log\log n)^{1/2}}$. This gives a constructive lower bound $R(k) > k^{\log k/\log\log k}$, still far short of solving the following problem by Erdős.

Problem 8($100 [CG98]) :
Give a constructive proof for $R(k) > (1+c)^n$ for any $c > 0$. In other words, for every n, construct a graph on n vertices which does not contain any complete subgraph of size $c \log n$ and does not contain any independent set of size $c \log n$ for some constants c.

The recent breakthrough of Mattheus and Verstraete [MV24] for $R(4,t)$ portends the trend of combining explicit constructions with randomization. By using a construction from algebraic geometry, called unitals, they construct an appropriate K_4-free graphs and then perform several rounds of randomization (e.g., using random block method to construct random K_4-free graphs, sampling via the method of containers and applying the ideas of random blowups to achieve the desired graph).

Throughout the literature in combinatorics, there are scattered many gems of explicit constructed graphs, many of which are derived by using tools in various areas of mathematics including number theory, algebra, projective geometry, coding theory, among others. Of particular interests are the expander

graphs constructed by Lubotzky, Phillips, and Sarnak [LPS88] that achieve the best possible eigenvalue bounds and therefore satisfy neighborhood expansion properties similar to the behavior of random sparse graphs. Another development is the family $\mathcal{Q}$ of quasi-random graphs [CGW89], which consists of many equivalent graph properties (as described in Section 3.1) that can be utilized to validate explicit constructions. We remark that the property of neighborhood expansion (i.e., the neighborhood $N(S)$ of S satisfies $|N(S)| \geq (1+c)|S|$ for any vertex subset S with less than half of the vertices, for some constant c,) does not belong to $\mathcal{Q}$. For example, replacing a subgraph on $n/2$ vertices of a random graph $G_{n,p}$ by a clique will result a graph satisfying vertex expansion property but does not satisfy properties in $\mathcal{Q}$. There are many other graph properties that are not in $\mathcal{Q}$ but is implied by $\mathcal{Q}$, such as having diameter small (like $\log n$). Of course, the Ramsey property $\mathcal{P}_{Ramsey}$ is not in $\mathcal{Q}$ either. The quasi-random class $\mathcal{Q}$ is just the starting point of the attempt to classify all graph properties.

The interplay of randomness and structures has led to powerful tools, one of which is Szemerédi's regularity lemma [Sze78] that asserts that, for any ε, any graph can be partitioned into finitely many parts so that almost all but εn^2 edges are contained in the union of quasi-random bipartite graphs between pairs of parts. (The number of parts depends on ε and independent of the size of the graph.) Therefore the regularity lemma can be used to approximate a dense graph by a finite structure so that the edges between each pair of parts are well-behaved via quasi-randomness. Among the numerous applications of the regularity lemma, the most notable is the role it plays in proof of Green and Tao [GT08] showing that the primes contain arbitrarily long arithmetic progressions. Although Szemerédi's regularity lemma is only effective for dense graphs, there is a version [Chu21] of regularity lemma for sparse graphs with nontrivial clustering coefficients. The clustering coefficient of a graph is the ratio of the number of triangles and the number of paths in the graph. The clustering effect, often described as "friends of friends are friends", is part of the so-called "small-world phenomenon", shared by many social and information networks. It is perhaps not surprising that many real world graphs can be viewed as combinations of (simple) structures and (controlled) randomness.

5 Van der Waerden numbers

Lastly, we return to the classical theorem of van der Waerden [vdW27] on arithmetic progressions that for any finite coloring of integers, there always exists arbitrarily long monochromatic arithmetic progressions. The finite version (for two colors) asserts the existence of the least number $W(n)$ such that if the integers $\{1,2,...,W(n)\}$ are 2-colored, then a monochromatic n-term arithmetic progression, denoted by n-AP, must always be formed. The estimation of the function $W(n)$ has challenged mathematicians ever since van der Waerden proved this result in 1927. The upper bound, due to van der Waerden, grew like the Ackermann function, and was not even primitive recursive (his proof was a double induction on n and the number of colors). In 1988 Shelah [She88] improved the upper bound to a function residing in the 5th level of the Grzegorczyk hierarchy (basically towers of towers). In 2001 Gowers [Gow01] settled a conjecture ($1000) of Ron Graham by showing:

$$W(n) < 2^{2^{2^{2^{2^{n+9}}}}}$$

The following conjecture is still open:

Problem 9($1000 [Gra08]) : Prove that for all n,

$$W(n) < 2^{n^2}.$$

The best lower bound, due to Berlecamp [Ber68] in 1968, is

$$W(n) \geq (n-1)2^{n-1} \quad \text{for } n \text{ prime.}$$

Acknowledgments

The author wishes to thank Joel Spencer, Jacques Verstraete and Staszek Radziszowski for valuable discussions and reminiscences. The title and parts of the introduction were essentially borrowed from the eloquent writings of Ron Graham [GR87].

References

[ACP+22] R. Ageron, P. Casteras, T. Pellerin, Y. Portella, A. Rimmel, and J. Tomasik. New lower bounds for schur and weak schur numbers, 2022. Preprint. http://arxiv.org/abs/2112.03175. 9

[AKS80] M. Ajtai, J. Komlós, and E. Szemerédi. A note on ramsey numbers. *Journal of Combinatorial Theory, Series A*, 29:354–360, 1980. 7, 8

[AM24] V. Angeltveit and B. D. McKay. $r(5,5) \leq 46$, 2024. Preprint. http://arxiv.org/abs/2409.15709. 3

[BDR88] I. Broere, D. Döman, and J. N. Ridley. The clique numbers and chromatic numbers of certain paley graphs. *Quaestiones Math.*, 11(1):91–93, 1988. 6

[Ber68] E. Berlekamp. A construction for partitions avoiding long arithmetic progressions. *Canad. Math. Bull.*, 11:409–414, 1968. 12

[BK21] T. Bohman and P. Keevash. Dynamic concentration of the triangle-free process. *Random Structures & Algorithms*, 58:221–293, 2021. 8

[Boh09] T. Bohman. The triangle-free process. *Adv. Math.*, 221:1653–1677, 2009. 8

[CG98] F. Chung and R. L. Graham. *Erdős on Graphs, His Legacy of Unsolved Problems*. A K Peters, Wellesley, Massachusetts, 1998. 9, 10

[CGMS23] M. Campos, S. Griffiths, R. Morris, and J. Sahasrabudhe. An exponential improvement for diagonal ramsey, 2023. Preprint. http://arxiv.org/abs/2303.09521. 5, 6

[CGW89] F. Chung, R. L. Graham, and R. M. Wilson. Quasi-random graphs. *Combinatorica*, 9:345–362, 1989. 5, 6, 11

[Chu73] F. Chung. On the ramsey numbers $n(3,3,...,3;2)$. *Discrete Math.*, 5:317–321, 1973. 9

[Chu74] F. Chung. On triangular and cyclic ramsey numbers with k colors. In *Graphs and Combinatorics*, number 406 in Lecture Notes, pages 236–242, New York, 1974. Springer-Verlag. 9

[Chu21] F. Chung. Regularity lemmas for clustering graphs. *Advances in Applied Math.*, 126:article 101961, 2021. 11

[CJMS25] M. Campos, M. Jenssen, M. Michelen, and J. Sahasrabudhe. A new lower bound for the ramsey numbers $r(3,t)$, 2025. Preprint. http://arxiv.org/abs/2505.13371. 8

[Con09] D. Conlon. A new upper bound for diagonal ramsey numbers. *Annals of Mathematics*, 170:941–960, 2009. 5, 6

[CW75] D. K. Ray Chaudhuri and R. M. Wilson. On t-designs. *Osaka J. Math.*, 12:735–744, 1975. 10

[Erd47] P. Erdős. Some remarks on the theory of graphs. *Bull. Amer. Math. Soc.*, 53:292–294, 1947. 3, 4, 10

[Erd61] P. Erdős. Graph theory and probability, ii. *Canad. J. Math.*, 13:346–352, 1961. 7

[ES35] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935. 2

[Exo89] G. Exoo. A lower bound for $r(5,5)$. *Journal of Graph Theory*, 13:97–98, 1989. 2

[FW81] P. Frankl and R. M. Wilson. Intersection theorems with geometric consequences. *Combinatorics*, 1:357–368, 1981. 10

[GG55] R. E. Greenwood and A. M. Gleason. Combinatorial relations and chromatic graphs. *Canad. J. Math.*, 7:1–7, 1955. 2, 9

[GNNW24] P. Gupta, N. Ndiaye, S. Norin, and L. Wei. Optimizing the cgms upper bound on ramsey numbers, 2024. Preprint. http://arxiv.org/abs/2407.19026. 4, 5, 6

[Gow01] W. T. Gowers. A new proof of szemerédi's theorem. *Geom. Funct. Anal.*, 11:465–588, 2001. 11

[GR87] R. L. Graham and V. Rödl. Numbers in ramsey theory. In C. Whitehead, editor, *Surveys in Combinatorics 1987 (New Cross, 1987)*, volume 123 of *London Math. Soc. Lecture Note Ser.*, pages 111–153. Cambridge Univ. Press, Cambridge, 1987. 5, 12

[GR90] S. Graham and C. Ringrose. Lower bounds for least quadratic non-residues. In *Analytic Number Theory: Proceedings of a Conference in Honor of Paul T. Bateman*, pages 269–309, 1990. 6

[Gra80] R. L. Graham. On partitions of $\mathbb{E}^n$. *Journal of Combinatorial Theory, Series A*, 28:89–97, 1980. 1

[Gra08] R. L. Graham. Old and new problems and results in ramsey theory. In E. Gyori, G. O. H. Katona, and L. Lovasz, editors, *Horizons of Combinatorics*, Bolyai Society Mathematical Studies, pages 105–118. 2008. 12

[GS90] R. L. Graham and J. H. Spencer. Ramsey theory. *Scientific American*, pages 112–117, July 1990. 3

[GT08] B. Green and T. Tao. The primes contain arbitrarily long arithmetic progressions. *Annals of Mathematics*, 167(2):481–547, 2008. 11

[HP21] B. Hanson and G. Petridis. Refined estimates concerning sumsets contained in the roots of unity. *Proc. Lond. Math. Soc.*, 122(3):353–358, 2021. 6

[Kal66] J. G. Kalbfleisch. *Chromatic Graphs and Ramsey's Theorem*. PhD thesis, University of Waterloo, January 1966. 2

[Kim95] J. H. Kim. The ramsey number $r(3,t)$ has order of magnitude $t^2/\log t$. *Random Structures and Algorithms*, 7:173–207, 1995. 7, 8

[LPS88] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261–277, 1988. 11

[Mon71] H. L. Montgomery. *Topics in Multiplicative Number Theory*, volume 227 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1971. 6

[MR97] B. D. McKay and S. P. Radziszowski. Subgraph counting identities and ramsey numbers. *Journal of Combinatorial Theory, Series B*, 69:193–209, 1997. 3

[MV24] S. Mattheus and J. Verstraete. The asymptotics of $r(4,t)$. *Annals of Mathematics*, 199(2):919–949, 2024. 8, 10

[PGM20] G. Fiz Pontiveros, S. Griffiths, and R. Morris. The triangle-free process and the ramsey number $r(3,k)$. *Mem. Amer. Math. Soc.*, 263(1274):v+125, 2020. 8

[Rad24] S. P. Radziszowski. Small ramsey numbers. *Electronic Journal of Combinatorics*, 2024. Dynamic survey #1. 3, 9

[Ram30] F. P. Ramsey. On a problem of formal logic. *Proc. London Math. Soc.*, 30:264–285, 1930. 1, 2

[Sah23] A. Sah. Diagonal ramsey via effective quasirandomness. *Duke Mathematical Journal*, 172(3):545–567, 2023. 5, 6

[She83] J. B. Shearer. A note on the independence number of triangle-free graphs. *Discrete Math.*, 46:83–87, 1983. 7, 8

[She88] S. Shelah. Primitive recursive bounds for van der waerden numbers. *J. Amer. Math. Soc.*, 1:683–697, 1988. 11

[Spe75] J. Spencer. Ramsey's theorem — a new lower bound. *J. Comb. Theory, Ser. A*, 18:108–115, 1975. 4, 5, 8

[Sze78] E. Szemerédi. Regular partitions of graphs. In J.-C. Bermond, J.-C. Fournier, M. Las Vergnas, and D. Sotteau, editors, *Proc. Colloque Inter. CNRS*, pages 399–401, 1978. 11

[Tho88] A. Thomason. An upper bound for some ramsey numbers. *J. Graph Theory*, 12:509–517, 1988. 5, 6

[vdW27] B. L. van der Waerden. Beweis einer baudetschen vermutung. *Nieuw Arch. Wisk.*, 15:212–216, 1927. 1, 11

[Vin54] I. M. Vinogradov. *Elements of Number Theory*. Dover, 1954. 6

[Wal71] K. Walker. An upper bound for the ramsey number $m(5,4)$. *Journal of Combinatorial Theory*, 11:1–10, 1971. 3

AUTHOR

Fan Chung

Univ of California, San Diego
San Diego, CA 92093
United States

fan@ucsd.edu
http://math.ucsd.edu/~fan