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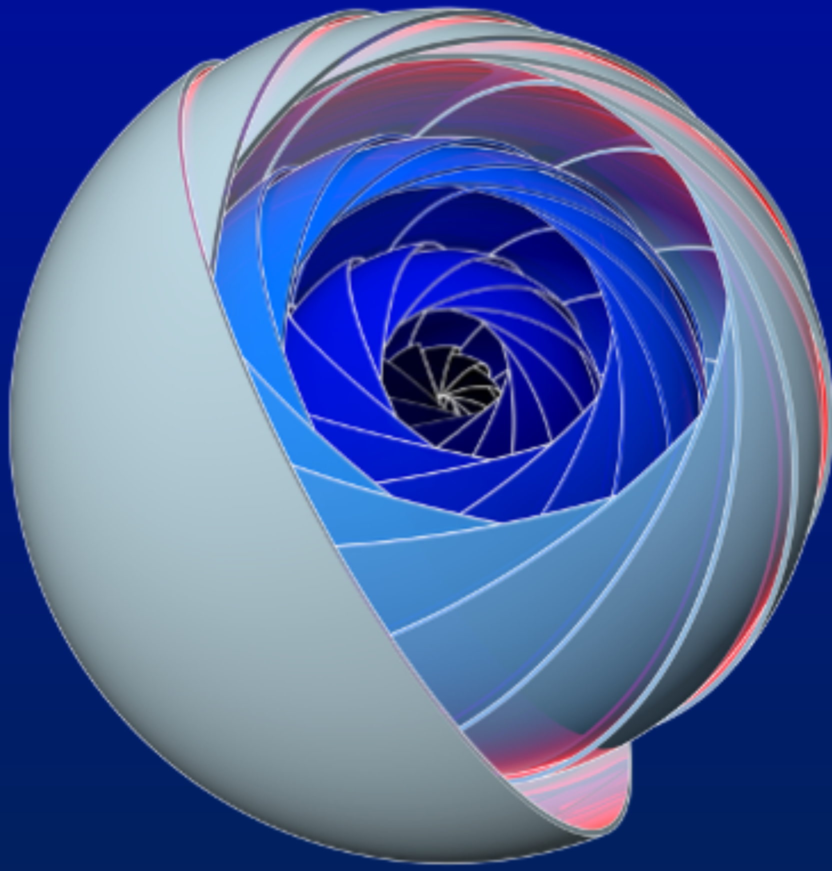

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Characterizing the 4-sphere, S^4

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Abstract:
We survey the status of the smooth 4-dimensional Poincaré Conjecture, one of the central open problems in mathematics.

Key words and phrases:
Smooth Poincare Conjecture; 4-Dimensional Manifolds; Smooth 4-Sphere; 4-Sphere Recognition

1 The smooth 4-dimensional Poincaré Conjecture

Once Perelman established the 3-dimensional Poincaré Conjecture [MT07] and Freedman proved the topological 4-dimensional Poincaré Conjecture [Fre82], the smooth 4-dimensional Poincaré Conjecture emerged as the central remaining piece of the puzzle, the holy grail of low-dimensional topology.

Conjecture 1.1. *If a closed, compact, smooth* 4*-manifold X^4 has the homotopy type of a* 4*-dimensional sphere, then X is diffeomorphic to S^4.*

Henri Poincaré laid the foundations of algebraic topology in two foundational papers [Poi04b, Poi04a] (see [Poi10] for an English translation with commentary), defining now ubiquitous terms such as *simply connected* and *homeomorphism*. He used invariants such as Betti numbers and torsion to distinguish manifolds (homology groups came later with Noether [Hil88]). He even introduced a forerunner of Morse theory by taking a smooth function on an n-manifold embedded in Euclidean space and observing how its level sets change when passing a critical point.

In his first paper, Poincaré wondered if a closed 3-manifold with no torsion and the same Betti numbers as S^3 was homeomorphic to S^3. In the latter paper he described a non-simply connected homology 3-sphere, the now famous Poincaré homology 3-sphere, with fundamental group isomorphic to the 120 element binary icosahedral group. Poincaré first described it by drawing two pairs of loops on the surface of a genus-2 surface. Gluing disks to each of the loops and then gluing in a 3-ball gave a description of the resulting 3-manifold. Heegaard later showed that any 3-manifold can be split into two handlebodies in a similar way. See Figure 1, from a 1907 paper of Dehn and Heegaard [DH07, Figure 3].

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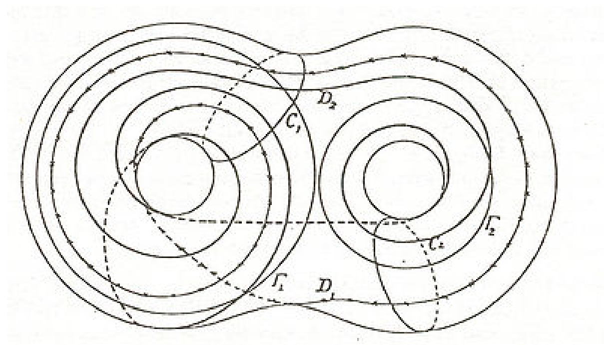

Figure 1: A Heegaard diagram for the Poincare homology sphere.

We now know many interesting ways of describing this manifold [KS]. One gives the Poincaré homology 3-sphere as the boundary of the 4-manifold obtained by adding a 2-handle to the 4-ball along the trefoil knot with framing $+1$. Another describes it as the quotient of S^3 by an action of the binary icosahedral group, a subgroup of $SO(4)$.

Many topologists spent years, sometimes lifetimes, trying to settle the 3-dimensional Poincaré conjecture before eventually Perelman proved it using geometric analysis [MT07]. In 1935 a notable attempt was made by JHC Whitehead, who erroneously claimed that a contractible open 3-manifold was homeomorphic to $\mathbb{R}^3$ [Whi35]. This led him to develop the notion of a space that is simply connected at infinity and to the *Whitehead manifold*, a contractible non-compact 3-manifold that is not homeomorphic to $\mathbb{R}^3$. The Whitehead manifold is the complement in S^3 of the Whitehead continuum, a set obtained via an infinite intersection process that in turn played a considerable role in Freedman's work [Fre82].

In 1956 Milnor discovered a 7-manifold that is homeomorphic but not diffeomorphic to the 7-sphere [Mil56]. Topologists were forcibly reminded that they had to pay attention to category, distinguishing between homeomorphism, PL-homeomorphism (in which two simplicial complexes are equivalent if they have isomorphic subdivisions), and diffeomorphism (usually meaning homeomorphic via C^∞ maps).

Milnor was interested in $(n-1)$-connected $2n$-manifolds, and the first interesting case, $n = 4$, was made with three handles of index $0, 4, 8$. After adding the 4-handle the boundary has to be a 7-sphere in order to add the final 8-handle. The 4-handle was added to an unknotted 3-sphere with framing given by an element of $\pi_3(SO(4)) = \mathbb{Z} \oplus \mathbb{Z}$. This element can be described using the quaternions; if q is a quaternion and p is a unit quaternion, then the map which sends $p \in S^3$ to the rotation sending q to $p^2qp^{-1} = p^2qp^{-2}p$ gives an 8-manifold with boundary a homotopy 7-sphere Σ (the two generators of $\pi_3(SO(4)) = \mathbb{Z} \oplus \mathbb{Z}$ are given by conjugation and by multiplication by a quaternion.)

Milnor knew that Σ could not be diffeomorphic to S^7 because after capping off with the 8-handle the resulting smooth, closed 8-manifold would have an impossible second Pontrjagin class.

At first he thought he had a counterexample to the *Generalized Poincaré Conjecture*, the extension of the 3-dimensional conjecture to higher dimensions. The generalized version states that a homotopy n-sphere is homeomorpic to S^n. But Milnor quickly constructed a smooth Morse function on Σ with only two critical points, a minimum and a maximum. So Σ was the union of two smooth balls, and thus homeomorphic to S^7 (this follows because a diffeomorphism of S^6 can be coned to a homeomorphism of B^7 or to a PL homeomorphism if the S^6 is given a smooth triangulation; note the philosophy that the difference between DIFF and PL or TOP is that one can take cones in the latter two, but not in the DIFF category.)

Thus Σ is a smooth manifold which is the union of two 7-balls that are glued together by an exotic diffeomorphism of S^6, one that cannot be smoothly isotopic to the identity. Note in contrast that if there is a counterexample to the smooth 4-dim Poincaré Conjecture, then it is the union of two 4-balls, one of which must be fake (not diffeomorphic to B^4) because there are no exotic diffeomorphisms of S^3 [Hat83].

Soon after Milnor's work, Smale [Sma61] came at the problem from an unexpected direction, settling the Generalized Poincaré Conjecture with his h-cobordism theorem. This had as a corollary that any smooth homotopy n-sphere is PL homeomorphic to S^n for $n \geq 5$. Within weeks of hearing of this, Stallings [Sta60] showed that any PL homotopy n-sphere is homeomorphic to S^n and a few years later Newman [New66] reached the same conclusion while only assuming that the homotopy n-sphere was a topological manifold. All these results were shown to hold for $n \geq 5$.

Dimensions three and four were still open. In 1981 Freedman [Fre82] announced a proof that any homotopy 4-sphere was homeomorphic to S^4 (and much more). The stumbling block in dimensions 4 and below was that a 2-handle could not necessarily be smoothly embedded in a situation where homotopy theory gave no obstructions (in contrast to higher dimensions, as explained in the next section). Casson had shown that where homotopy theory allowed for a 2-handle, a proper homotopy equivalent $B^2 \times \mathbb{R}^2$ could be embedded. Freedman showed that this *Casson handle* contained a topological 2-handle, homeomorphic to $B^2 \times \mathbb{R}^2$. He required that the 4-manifold fundamental group was polycyclic, meaning that it did not grow too fast compared to the number of generators (e.g. not the free product of two infinite cyclic groups). (It is not known whether Casson handles contain topological 2-handles without constraints on the fundamental group of a 4-manifold.) With this result about 2-handles, Freedman constructed topological 4-manifolds without any smooth (or PL) structure.

Within six months of Freedman's work, Donaldson [Don83] applied gauge theory to construct restrictions on the possible cohomology structure of smooth 4-manifolds. Then Friedman and Morgan [FM88] and independently Okonek and Van de Ven [OVdV86] showed that the 4-manifolds obtained by two log transforms with multiplicities p and q on the elliptic surface $\mathbb{C}P^2$ with nine points blown up were homeomorphic but not diffeomorphic, for infinitely many pairs p and q.

One example is constructed by adding a 2-handle with framing-one to a trefoil knot lying in the boundary of B^4, giving a 4-manifold X. The boundary of this 4-manifold is the Poincaré homology 3-sphere. Freedman showed that any homology 3-sphere bounds a contractible, topological 4-manifold, so the Poincare homology 3-sphere bounds a contractible 4-manifold Y. Gluing together X and Y gives a 4-manifold Z homeomorphic to $\mathbb{C}P^2$, which Freedman named the *Chern manifold*, in honor of SS Chern. The Chern manifold cannot have any smooth structure by an argument with the Kirby-Siebenmann invariant applied to $Z \times S^1$ [Fre82, Theorem 1.5].

As of June 2025 the following conjecture is still open.

Conjecture 1.2. *Each compact, closed and smoothable topological* 4*-manifold admits a countably infinite number of smooth structures, and each non-compact* 4*-manifold admits uncountably many.*

Four space $\mathbb{R}^4$ itself admits a one-parameter family of distinct smooth structures. This is because an early construction [Kir06] of an exotic smooth structure on R^4 has the property that there exists a compact set $K \subset \mathbb{R}^4$ such that there are no smooth 3-spheres outside K, although there are topological 3-spheres enclosing ever larger open balls. As open sets, each of these open balls inherits a smooth structure. Taubes' periodic ends theorem [Tau87] shows that no two of these can be diffeomorphic, since if a smooth open 4-manifold has a periodic end ($M^3 \times [0,1)$) then the open 4-manifold has a smooth boundary, which would be S^3 in our case. Even more strangely, the standard $\mathbb{R}^4$ contains an uncountable collection of open submanifolds that are distinct exotic $\mathbb{R}^4s$ [DF92].

This was surprising, since Stallings [Sta62] had shown that $\mathbb{R}^n$ has a unique smooth structure for all $n \neq 4$ by a clever, but simple, engulfing argument in the PL category. (Note that PL and Smooth structures on $\mathbb{R}^n$ coincide.)

In contrast, there are only a countable number of compact n-manifolds in any dimension [CK70], up to homeomorphism. If not, we fix n and can assume that all of our n-manifolds are embedded in a fixed high-dimensional Euclidean space from which they inherit a metric. Each manifold can be covered by finitely many coordinate charts, so there must be a number k for which there are uncountably many n-manifolds with each covered by no more than k coordinate charts. Then there must be a pair of manifolds whose coordinate charts are very close to each other. Then by the local contractibility of homeomorphisms [EK71] the charts are topologically isotopic and thus the two manifolds are homeomorphic.

Remark. The smooth structure on S^n is known to be unique for $n = 1,2,3,5,6,12$, 56, 61. A list of all dimensions with this property, and thus for which the smooth Poincaré Conjecture holds, is not known. In this sense the general smooth Poincaré Conjecture remains open in high dimensions, as well as in dimension four.

2 The h-cobordism Theorem

The h-cobordism theorem states the following:

Theorem 2.1. *Let W be a smooth, compact $(m+1)$-dimensional manifold with two m-dimensional boundary components, M_0 and M_1. Suppose that W is simply connected, $m \geq 5$, and the inclusions $M_i \hookrightarrow W$, induce homotopy equivalences for $i = 0,1$. Then W is diffeomorphic to $M_i \times [0,1]$, for $i = 0$ or* 1.

An equivalent formulation assumes that W is simply connected, $m \geq 5$, and each of M_0 and M_1 is a deformation retract of W.

Corollary 2.2. *A smooth homotopy sphere Σ of dimension $n \geq 6$ is PL homeomorphic to S^n.*

Proof. Remove two open n-balls from Σ, obtaining an h-cobordism with boundary diffeomorphic to two copies of S^{n-1}. The h-cobordism Theorem implies that this is diffeomorphic to a product, $S^{n-1} \times [0,1]$. We can glue an n-ball to each of the two boundary components, hoping to get an n-sphere. But we have to be careful, as while the two boundary components are diffeomorphic to S^{n-1}, we do not know that the two glueing diffeomorphisms extend to give a diffeomorphism from Σ to S^n. If we just glue one n-ball to the h-cobordism then we are adding a collar to an n-ball, which does yield an n-ball. We can conclude that Σ is diffeomorphic to the union of two smooth n-balls, glued together by some diffeomorphism g. It does not follow that Σ is diffeomorphic to S^n, as g may not be smoothly isotopic to the identity (or even smoothly pseudo-isotopic). However, we can always construct a homeomorphism (in fact a PL homeomorphism) from Σ to S^n by coning g. □

Remark: Milnor's exotic smooth structure on S^7 arose because he was able to show that his homotopy 7-sphere Σ was the union of two 7-balls, but the gluing diffeomorphism $g : S^6 \to S^6$ did not extend over the 7-ball. A description of Σ^7 is not hard; it is the boundary of the eight-dimensional manifold obtained by plumbing tangent disk bundles of S^4 using the **E8** diagram. A construction of the exotic diffeomorphism g is not known.

Question 2.3. *Can the exotic gluing diffeomorphism $g : S^6 \to S^6$ be explicitly described?*

Proof of the h-cobordism theorem:

We outline the proof of this important result. A full proof is in [M+65].

2.1 Handles

We begin with a Morse function $f : W \to [0,1]$ with the property that $f^{-1}(0) = M_0$ and $f^{-1}(1) = M_1$. A Morse function has isolated critical points p where $df(p) = 0$ and these are non-degenerate, so that each p has a coordinate neighborhood with p at the origin and such that

$$f(x_1, x_2, \ldots, x_{m+1}) = f(p) - x_1^2 - x_2^2 \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{m+1}^2.$$

The number k is called the *index* of the critical point.

When W has dimension one, the critical points are either minima or maxima, and not so interesting, but in dimension two there are also saddle points ($k = 1$) which give rise to descending and ascending 1-cells.

By thickening the descending k-cell using the coordinate chart, we obtain a k-handle, $B^k \times B^{m+1-k}$, with the descending k-cell given by $B^k \times 0$. If we consider the structure of the manifolds $M_y = f^{-1}(-\infty, y)$, we can consider how these change as we pass through a critical point. An analysis shows that passing through a critical point of index k corresponds to adding a k-handle. Namely if there is an index-k critical point at height c, then $M_{c+\varepsilon}$ is obtained from $M_{c-\varepsilon}$ by adding a k-handle. In describing a k-handle in an n-manifold we write $B^n = B^k \times B^{n-k}$ and then specify how to glue the portion of the boundary given by $S^{k-1} \times B^{n-k}$ to the level set $M_{c+\varepsilon}$. Such a gluing, in the orientable case, is determined up to istopy by an element of $\pi_{k-1}(SO(n-k))$.

The proof of the h-cobordism theory uses the assumed homotopy-theoretic conditions to modify f to remove all interior critical points, or equivalently all handles. The homotopy type gives conditions on how $(k+1)$-handles run algebraically over the k-handles, data which determines the homology.

If f has no critical points, then df is never zero, so the descending gradient vector field always points down and can be integrated to give a flow from M_1 to M_0. This flow then gives the desired product structure on W.

A critical point of index $k+1$ and one of index k will cancel if their corresponding descending $k+1$-cell and ascending $(m+1-k)$-cell intersect in an arc joining the two critical points. Equivalently, the descending and ascending cells meet at a single point in an intermediate level set of f. The prototype cancellation example (for a one-dimensional manifold) is the one-parameter family $f_t : \mathbb{R} \to \mathbb{R}$ given by $f_t(x) = x^3 + tx,\ -1 \le t \le 1$. For $t = -1$ we see a local maximum ($k = 1$) at $x = -1$ and a local minimum ($k = 0$) at $x = 1$ and these two cancel when $t = 0$. This generalizes to all dimensions; the key is to get the ascending and descending manifolds to intersect geometrically at a single point in an intermediate level.

We can visualize this cancellation on a surface by drawing a 2-dimensional neighborhood of the cancelling arc, filled with small arrows indicating the gradient vector field. The series of diagrams in Figure 2 shows how this gradient vector field changes as t goes from -1 to 1.

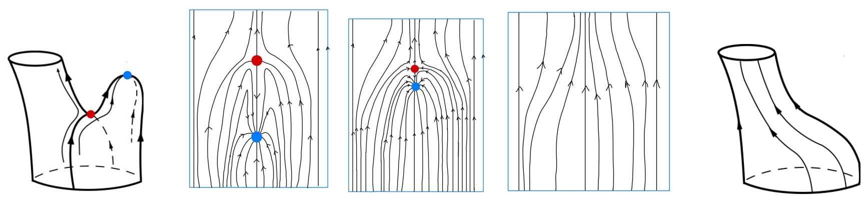

Figure 2: A cancelling pair of critical points of index 0 and index 1 and their corresponding gradient vector fields.

Another phenomenon occurs when the descending k-cell of an index k critical point hits another critical point of index k. This will not happen for a generic Morse function, but will occur when we consider one-parameter families of Morse functions. These phenomena were studied by Cerf [Cer70]. A natural one-parameter perturbation sends the descending k cell to one side or the other of the lower index-k critical point. It looks like (and is called) one cell (or handle) sliding over another, as in Figures 3 and 4.

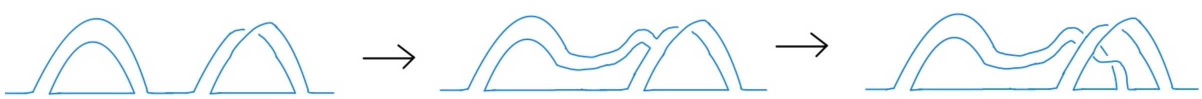

Figure 3: When a handle slides over a second handle, the attaching sphere adds a copy of the second attaching sphere and the framing of the handle it slides over.

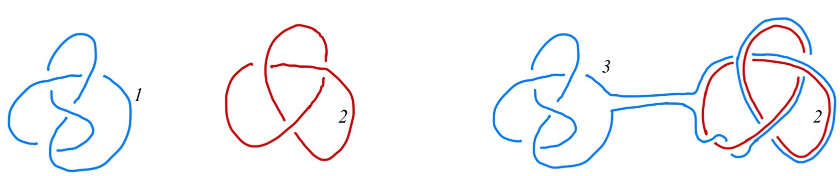

Figure 4: A 2-handle attached to the 3-sphere along a figure eight knot slides over a second handle attached along a trefoil knot. The numbers indicate the framings before and after the slide.

At a level set of f just below a critical point of index k, we see an attaching $(k-1)$-sphere, to which a descending k-handle is attached. When the Morse function is altered so that this k-handle slides over a lower k-handle, we see the attaching sphere move close to the other attaching sphere and then "suddenly" pop over. The new attaching $(k-1)$-sphere has been altered by a connected sum with the lower attaching $(k-1)$-sphere.

2.2 Algebraic Analogues

A Morse function on a manifold gives rise to cellular homology groups. Given $f : W \to R$, define the kth chain group C_k to be the free abelian group generated by the critical points of index k, and boundary maps $\partial_k : C_k \to C_{k-1}$ by counting with sign the number of intersection points of a descending k-cell with an ascending $(m+1-k)$-cell in a middle level, or equivalently counting with sign the number of descending arcs that hit the $(k-1)$-index critical point. This gives a chain complex for which $\partial_{k-1}\partial_k = 0$, and the kth homology group H_k is the quotient of the kernel of $\partial : C_k \to C_{k-1}$ by the image of $\partial : C_{k+1} \to C_k$.

Once bases for the chain groups are chosen, the boundary homomorphisms are represented by integer matrices. The bases are chosen according to the critical points of the Morse function. If a descending k-cell α slides over another k-cell β, then the basis changes by sending (α,β) to $(\alpha,\alpha+\beta)$, or in terms of matrices, adding column α to column β. Similarly sliding a $k-1$ cell over another corresponds to adding one row to another.

By such row and column operations, each matrix for each k can be reduced to an identity submatrix surrounded by zeroes.

Each column in the identity submatrix corresponds to a k-handle that algebraically cancels a $(k-1)$-handle, meaning that a descending $(k-1)$-sphere hits an ascending $(m-k)$-sphere (in an m-dimensional level set) algebraically once (intersection points have signs because in choosing a generator for a $\mathbb{Z}$ in a chain group, we have implicitly chosen an orientation for the descending cell, and homology is independent of these choices).

We want to cancel pairs of intersection points of opposite signs, reducing to the case where the intersection consists of a single point, so that the k-handle and the $(k-1)$-handle geometrically cancel. See Figure 2. To carry out this cancellation we use the Whitney trick, which requires dimension $m+1 \geq 6$.

2.3 The Whitney trick

Whitney showed that a smooth m-dimensional manifold M embeds in $\mathbb{R}^{2m+1}$[Whi44] and later improved this to show that any manifold M embeds in $\mathbb{R}^{2m}$ [Whi92]. He did this by removing pairs of double points with oppositely signed orientations. This method became known as the *Whitney trick*. (An immersion of M in $\mathbb{R}^{2m}$ may have self-intersection $r \neq 0$. To apply the Whitney trick, r points of oppositely signed self-intersection are introduced, using the higher-dimensional analogue of taking an arc in the xy-plane and adding a double point by connect summing to r copies of a figure eight, appropriately oriented. In $\mathbb{R}^{2m}$ a figure eight can be replaced by an m-sphere with a single self-intersection point. Such a sphere can be constructed by taking two unit m-disks lying in complementary m-planes of $\mathbb{R}^{2m}$ that intersect at the origin in $\mathbb{R}^{2m}$, one in $\mathbb{R}^m \times 0$ and the other in $0 \times \mathbb{R}^m$. The boundary of these two disks consists of a pair of linking $(m-1)$-spheres in S^{2m-1}. We can embed an $S^{m-1} \times I$ into S^{2m-1} bounded by these two S^{m-1} spheres, giving an immersion $S^m \to \mathbb{R}^{2m}$ with a single double point. Appropriately orienting the disks will lead to a positive or negative self-intersection.

A model showing how the Whitney trick eliminates pairs of intersection points is given by an arc in the xy plane that intersects the yz plane at two points, p and q. A disk D is bounded by the arc and the line segment between p and q, as in Figure 5.

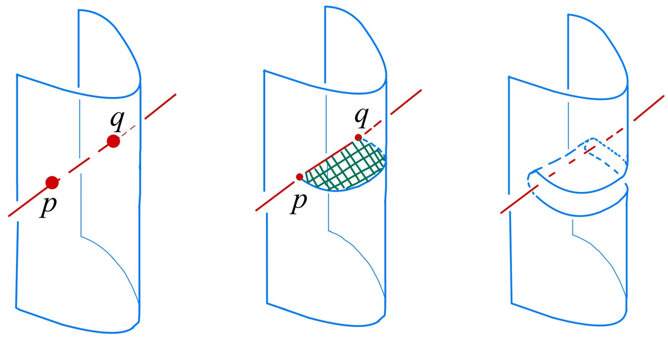

Figure 5: The Whitney trick

The model sits in $\mathbb{R}^3$ and it is easy to see that either the arc can be pushed across D removing the intersection points, or the plane can be pushed left to avoid the arc, again using D.

Suppose M^m and N^n are smoothly embedded manifolds in Q^q, where $m+n=q$, all are oriented, and they intersect transversely with algebraic intersection zero. Let p and q be intersection points, oriented in opposite directions, which we wish to eliminate, on the way to making M and N disjoint. Assuming that M and N are connected, we can join p and q by arcs in M and N. Then we want the union of the two arcs, the Whitney circle, to bound a smoothly embedded D. This is possible if Q is simply connected and $q \geq 5$, for simple connectivity implies that D can be mapped in and dimension 5 or more means that double points in D can be removed (this is the point at which there is so much difficulty extending the argument to dimension four). In addition, the normal bundle to D in Q must contain the normal bundle to the arcs in M and in N. Finally, the Whitney sum of the oriented tangent bundles to M and N must give opposite orientations for Q at p and q.

The conditions in the h-cobordism theorem are taylored to allow the Whitney trick to be used to cancel critical points, and we can then complete the proof of the theorem. □

3 Bushel baskets of possible counterexamples

In his famous book *A quick trip through knot theory,* Fox comments that there are “bushel baskets” of possible counterexamples to the 3-dimensional Poincaré Conjecture. Fox was thinking of irregular 3-fold branched covers of knots in S^3, but not long after Burde [Bur71] showed that all of these were, in fact, diffeomorphic to S^3. After that, there were no obvious families of potential counterexamples in dimension three.

Since the 1950s we have known of at least two bushel baskets of possible counterexamples to the 4-dimensional Poincaré conjecture.

3.1 Gluck twists

The Gluck twist [Glu62] removes an $S^2 \times B^2$ from S^4 and sews it back in by an interesting diffeomorphism of its boundary, $S^2 \times S^1$. There is only one non-trivial diffeomorphism up to isotopy, one that fixes the north and south poles and rotates the latitude $S^2 \times \theta$ by $\theta \in [0, 2\pi]$. This diffeomorphism generates $\pi_1(SO(3)) \approx \mathbb{Z}_2$. This always gives a homotopy 4-sphere, but is only known to be S^4 in special cases (see the forthcoming problem list, K3 [BR$^+$].)

3.2 Approaches connected to the Andrews-Curtis Conjecture

A second basket of examples comes from homotopy 4-spheres created from non-trivial presentations of the trivial group. For example, consider the following presentation of the trivial group:

$$\langle x, y \mid xyx^{-1} = y^{-1}xy,\ x^4 = y^5 \rangle \tag{1}$$

This can be seen to represent the trivial group by raising the first relation to the fifth power, substituting x^4 for y^5 and simplifying.

If one builds a homotopy 5-ball by adding 1-handles to B^4 for the two generators, and 2-handles for the two relations, then the boundary is a smooth homotopy 4-sphere. The fundamental group in this case has a balanced presentation with the same number of generators and relations. There is no obvious reason why these manifolds should be diffeomorphic to the standard S^4. One can try to simplify the homotopy 5-ball by handle slides which correspond to the Tietze moves in the algebraic reduction of the presentation. The difficulty is that when raising the first relation to the fifth power, one must slide the 2-handle over itself five times. That cannot be done without a second copy of the 2-handle to slide over. Adding a second copy of the 2-handle without changing the 5-manifold can be done by adding a $(2-3)$-cancelling pair of handles, but that introduces 3-handles which cause different problems.

Gompf [Gom91] was able to add a cancelling $(2-3)$-pair to Akbulut's handlebody diagram ([AK85], Fig.29) and show that the resulting homotopy 4-sphere is indeed S^4. But this proof did not show that Equation (1) can be reduced to the trivial presentation by Andrews-Curtis moves.

The rough description of Andrews-Curtis for topologists is that the only Tietze moves that can be used are ones that do not involve remembering past relations. These are the Nielsen moves on the presentation, a subset of the Tietze moves that maintains a balanced presentation, along with arbitrary conjugations. Just as we only have two 2-handles at any one time, we are not allowed to have more than two relations at any one time. The Andrews-Curtis Conjecture [AC65, Met85] states that any balanced presentation of the trivial group can be reduced to the trivial presentation without remembering, i.e. using only Nielsen moves. If true, it would imply that many of the potentially exotic smooth-homotopy-spheres constructed as above are standard 4-spheres. As of today the Andrews-Curtis Conjecture appears unlikely to hold. The analogous statement is false if the trivial group is replaced by the group of order two. In addition, Bridson has constructed simple presentations that are only known to be trivialisable by arbitrarily large numbers of Tietze moves [Bri].

In summary, if the Andrews-Curtis Conjecture holds then a large family of smooth homotopy 4-spheres would be known to be standard. However, a proof of the conjecture remains elusive.

3.3 Approaches via corks

Suppose that W^5 is a smooth h-cobordism between simply connected 4-manifolds X_0 and X_1. As in our sketch of the proof of the h-cobordism theorem above, we cancel all but the 2- and 3-handles in a handle decomposition of W. In a middle level of W, called X, we see 2-spheres $\{S_i\}$ that come from the descending 3-disks of the critical points of index 3, as well as 2-spheres $\{T_j\}$ coming from the ascending 3-disks of index 2 critical points.

After handle slides, we can assume that the algebraic intersection matrix $S_i \cdot T_j = \delta_{ij}$ is the identity matrix. That is, each descending 2-sphere intersects one ascending 2-sphere (in X) geometrically at $2k+1$ points, but algebraically once, and intersects any other descending 2-sphere algebraically zero times, as in Figure 6.

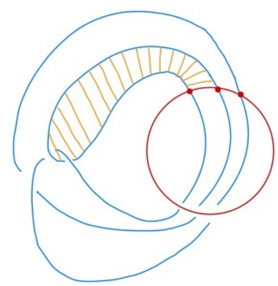

Figure 6: A red ascending 2-sphere intersects a blue descending 2-sphere geometrically in 3 points, but algebraically in one.

Unfortunately the Whitney trick is not available to remove pairs of intersection points, since the dimension of X is four.

Progress was made in describing the topology in a neighborhood of these 2-spheres in [CFHS96, Kir96]. The construction of a cork starts with choosing a base point in X, connecting it by disjoint arcs to base points in each sphere of S and T, and then thickening this tree of arcs to form a 0-handle. From a point of intersection of a sphere in S and a sphere in T, we add an arc to the base point of each sphere, and the union of these two arcs forms a 1-handle attached to the 0-handle, after thickening. After cutting these 1-handles out, the remains of the 2-spheres in S and T are disks, which when thickened form 2-handles which are attached to form Y. There are other 2-handles in X which cancel the 1-handles in Y, and these are then added to Y. After thickening Y to dimension 5, it is enlarged by adding the 2- and 3-handles of W to form the cork A. This cork is a subcobordism of W, with $A_i = A \cap X_i, i = 0, 1$.

The construction has the following properties:

1. The complement of A is a simply connected product cobordism, $W - A = (X_i - A_i) \times I, i = 0, 1$
2. $A = A_i \times I = B^5$.
3. The diffeomorphism in (2) between A_0 and A_1 can be chosen to be an involution on the boundary.

A_i is called a cork, for X_0 can be changed to X_1 by cutting out A_0 and gluing it back in by the involution. Any homotopy 4-sphere can be obtained from S^4 by cutting out a cork and regluing it.

The first example of a cork, predating [CFHS96], was found by Akbulut [Akb91b] when he discovered a cork that gave two distinct smooth structures on the K3 surface blown up once. This cork was the smallest known 4-manifold with an exotic structure rel boundary. it was later shown to be exotic absolutely [Akb91a]. The same four-manifold originally appeared in a paper of Barry Mazur [Maz61] in which he constructed the first examples of contractible smooth 4-manifolds with homology-sphere boundaries, which later became known as *Mazur manifolds*.

3.4 Approaches based on exotic RP^4s

A third bushel basket appeared in the 1970s when Cappell and Shaneson constructed exotic $\mathbb{R}P^4$s [CS76]. The double covers of these smooth manifolds gave potential counterexamples. Gompf [Gom91] showed that the simplest construction in [CS76] is actually covered by a standard S^4. This implies the existence of an exotic involution of S^4. An earlier example of an exotic involution on S^4 was given by Fintushel and Stern [FS81]. Since then many other Cappell-Shaneson exotic RP^4s have been shown to be covered by standard S^4s, but not all [FS81].

4 Relation to the smooth Schoenflies Conjecture

Let S^3 be the equator of S^4. The Schoenflies Conjecture states:

Conjecture 4.1. *Any smooth embedding of S^3 into S^4 bounds a smooth 4-ball.*

Note that if the embedding bounds a standard 4-ball on one side, then that ball can be shrunk to a small round ball whose complement in S^4 is also a standard 4-ball. An equivalent statement to the conjecture is that a smooth embedding of an equitorial S^3 into S^4 extends to a diffeomorphism from S^4 to S^4.

The conjecture is true in the topological category, as proved by Brown [Bro60], Mazur [Maz59] and Morse [Mor60] around 1960.

Lemma 4.1. *The smooth Schoenflies Conjecture is true if the smooth Poincaré Conjecture is true.*

Proof. Let B and B' be the clsoures of the two regions forming the complement of the embedded 3-sphere. Replace B' by a standard B^4, obtaining a homotopy 4-sphere which by assumption is S^4. But the complement of a smooth 4-ball in S^4 is also a smooth 4-ball, showing that B is standard. Then B' is also standard. □

The converse would be true if the following question had an affirmative answer:

Question 4.2. *Does every smooth homotopy 4-ball with S^3 boundary smoothly embed in S^4?*

Budney and Gabai [BG] have made progress towards disproving the smooth Schoenflies Conjecture by constructing smooth embeddings of the 3-ball into $S^1 \times S^3$ with fixed bounding S^2 that are not smoothly isotopic by an isotopy that fixes the boundary pointwise. Unfortunately, the non-simple connectedness of $S^1 \times S^3$ is vital to their argument.

5 Recognizing the 4-sphere

Conjecture 5.1. *There is no algorithm to recognize S^4.*

In the 1950s Markov [Mar60] proved that there could be no algorithm to distinguish all pairs of closed, compact, smooth 4-manifolds. His proof started with the fact that there is no algorithm to decide whether a finitely presented group G is trivial, proved by Novikov [Nov58] and Boone [Boo59] (also see [Man77] and [Gor95]).

The most natural setting for analyzing the complexity of manifold is the PL category. The complexity of a manifold is then measured by the number of simplices used to construct it. Note that there is a 1-1 correspondence between smooth and Pl manifolds in dimension four, and there are procedures that take a manifold in one category and describe it in the second category. We state the results below in the smooth category.

Definition 5.1. *A manifold X belonging to a collection $\mathcal{C}$ of manifolds is said to be* recognizable *if there exists an algorithm which can decide whether or not any manifold Y in $\mathcal{C}$ is diffeomorphic to X.*

Theorem 5.2. *The 4-manifold $X^4 = \#^k S^2 \times S^2$, for some $k \in \mathbb{Z}_+$, is not recognizable in the collection C of all smooth, compact, closed 4-manifolds.*

Gordon has shown that the value $k = 12$ suffices [Gor22]. This has recently been improved to $k = 9$ by Tancer [Tan23].

Corollary 5.3. *An algorithm to distinguish compact, smooth, closed 4-manifolds does not exist.*

The proof reduces Theorem 5.2 to the result of Boone and Novikov that there is no algorithm to recognize the trivial group within the collection of finitely presented groups $\mathcal{G}$ [Boo59, Nov58].

Lemma 5.4. *Given a finitely presented group G, we can construct a smooth 5-manifold Y_G^5 such that $\pi_1(Y_G) = \pi_1(\partial Y_G) = G$ and Y_G has no handles of index larger than 2.*

Proof. Suppose G has g generators $g_1, \dots g_g$ and r relations $r_1, \dots r_r$. We can construct a 5-manifold Y_G by adding g 1-handles to B^5 (obtaining $\#^g(S^1 \times B^4)$) and then adding r 2-handles according to the relations $r_1, \dots r_r$. Because the attaching circles of the 2-handles lie in a 4-manifold, these isotopy classes are determined by their homotopy classes in $\#^g S^1 \times S^3$, which in turn are determined by the relations.

A 2-handle $B^2 \times B^3$ is attached by an embedding of $S^1 \times B^3$. Given the embedding of $S^1 \times 0$, there are then two ways to extend that embedding, since $\pi_1(SO(3)) = \mathbb{Z}/2$. For example for the unknot in ∂B^5, the two framings give either $S^2 \times B^3$ or the non-trivial B^3 bundle over S^2, $S^2 \tilde{\times} B^3$. Let Y_G be defined by a choice of framings for the 2-handles.

The map induced by inclusion, $\pi_1(\partial Y_G) \to \pi_1(Y_G)$ is an isomorphism. It is an epimorphism because in a 5-dimensional manifold, transversality implies that there is enough room for any loop in Y_G to be pushed off the 2-complex spine of Y_G and thus into ∂Y_G. It is a monomorphism for the same reason; if a loop in ∂Y_G is homotopically trivial in Y_G, then that 2-disk can also be pushed off the spine and into ∂Y_G. ∎

Now define Y_g to be Y_G with g trivial 2-handles added, attached to the unlink in a small 4-ball in ∂B^5 which avoids all other attaching maps, and with the framing on each attaching circles that gives $B^2 \times B^3$. Note that $Y_g = Y_G \#^g S^2 \times B^3$.

The proof of the Theorem follows from the next Lemma.

Lemma 5.5. *With appropriate framings, G is trivial* $\iff \partial Y_g = \#^r S^2 \times S^2$.

Proof. The *if* part follows from $G = \pi_1(Y_g) = \pi_1(\partial Y_G) = 0$.

For the *only if* part, note that $\pi_1(\partial Y_G) = G = 0$. Then the attaching circles of the g trivial 2-handles can be homotoped and thus isotoped in ∂Y_G, to geometrically cancel the g 1-handles. (Note that we could not do this with the original r 2-handles because they do not lie in a simply connected 4-manifold, but rather in a connected sum of $S^1 \times S^3$s, whereas the extra g 2-handles may be thought of as added to ∂Y_G.) Do the cancellation, and what remains of Y_g is an unlink of r components to which are attached r 2-handles, so that Y_g is diffeomorphic to $\#^r S^2 \times B^3$ with boundary $\#^r S^2 \times S^2$ or a similar manifold in which one or more of the handles is added with a twist. However, going back to the original r 2-handles, we could have chosen their framings so that we avoid the twisted case. $\square$

Remark. We can start the above construction with $S^2 \tilde{\times} S^2$ rather than S^4. In this case the framing of the attached 2-handles does not affect the resulting four-manifold. There are two S^2 bundles over S^2, classified by $\pi_1(SO(3)) \approx \mathbb{Z}_2$. The manifold constructed by taking a connect sum of such bundles is diffeomorphic to either a connect sum of trivial bundles or a connect sum of twisted bundles, and the former occurs only if each summand is trivial. The proof of this fact is much like the classification of non-orientable surfaces. We refer to a connect sum of S^2 bundles over S^2, with at least one of the bundles twisted, as $\#^k S^2 \tilde{\times} S^2$. The final manifold is diffeomorphic to a connect sum of k copies of some combination of $S^2 \times S^2$s and $S^2 \tilde{\times} S^2$s, with at least one twisted summand. A modified Lemma 5.5 would then state that, irrespective of framings, we end up with $\#^{r+1} S^2 \tilde{\times} S^2$ if and only if G is trivial.

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